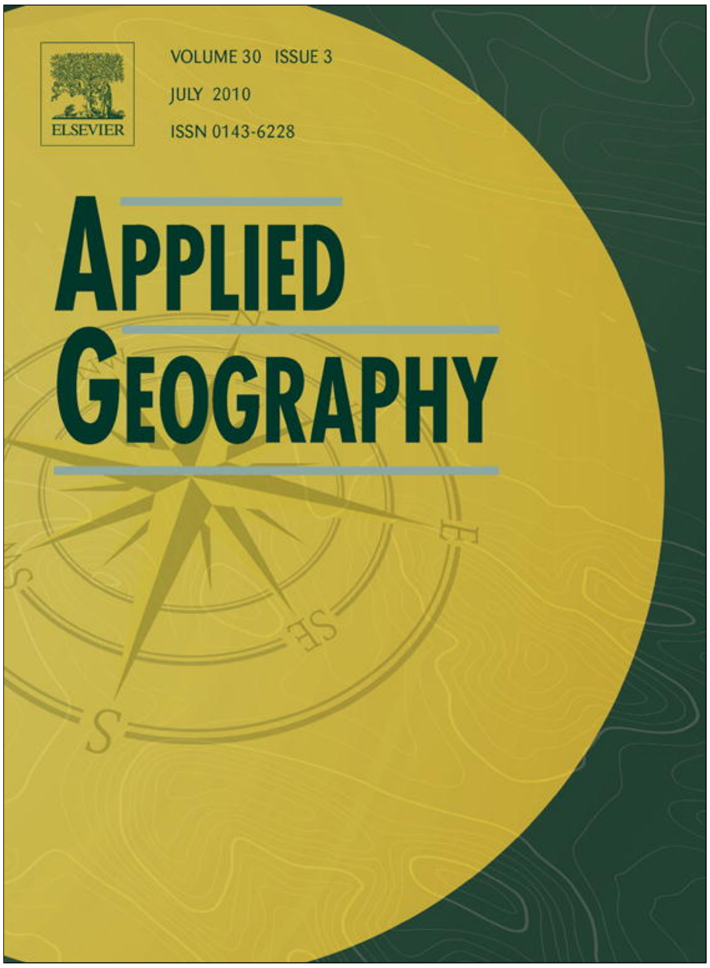

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Spatially explicit inverse modeling for urban planning

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A B S T R A C T

Urban modeling methods have traditionally followed a forward modeling approach. That is, they use data from today's situation to forecast or simulate future states of an urban system. In this paper, we propose an inverse modeling approach by which we shift our attention from solely forecasting or simulating future states of an urban system to steering it to a desired state in the future via key variables characterizing the system in the present. We first present a theoretical framework for the use of the inverse approach in urban planning. We test the power of the proposed method using a hedonic house price model in a metropolitan area in Switzerland to investigate the negative effects of densification on house prices. The model is calibrated by mixed geographically weighted regression in order to account for spatial variability of both key variables and model outputs. We show how devaluation of house prices caused by densification can be compensated by different levels of socioeconomic, locational as well as structural variables. We illustrate and discuss how trade-offs between variables may lead to more feasible results from an urban planning perspective. We conclude that the proposed method might be valuable for urban planners for developing implementable spatial plans based on future visions. In particular, the fact that other model specifications than hedonic house price model can also be employed to formulate an inverse model application, allows planners to address other type of problems or externalities from urbanization processes such as urban sprawls, environmental pollution or land uses change.

Introduction

Urban systems are complex systems made of a number of individual components that interact with one another through an intricate network (Baynes, 2009; Bretagnolle, Daudé, & Pumain, 2003; Liu, 2008) As Liu (2008) argues, urban systems consist of a set of elements or subsystems, such as population, land, employment, services and transport, to mention a few. All components of the system are interacting with each other through social, economic, and spatial mechanisms while they are also interacting with components of the environment. Some components such as urban population are expected to increase extensively over the upcoming decades. According to United Nations (2009), more than 50% of the population lives now in urban areas and this is expected to rise to 70% in 2050. Such rapid increase in the urban population will most likely cause people's welfare to decrease even long before 2050. Higher levels of population density in cities are generally associated with negative externalities such as pollution, traffic congestion and crime, among others, as well as with economic disequilibrium in the land and housing market. Yet, while planners are aware of these rapid changes, adaptation strategies and approaches tackling these growing challenges are still lacking.

A number of mathematical methods in the literature deal with urban development. The most popular are urban-growth logistic regressions which attempt to examine and forecast urban-growth using an econometric formulation (e.g. Allen & Lu, 2003; Hu & Lo, 2007; Landis & Zhang, 1998), neural-networks modeling by which the interaction between the different elements of an urban system is studied based on the way biological neural systems develop (e.g. Maithani, Jain, & Arora, 2007; Ou, Zhang, Ren, & Yao, 2003; Pijanowski, Brown, Shellito, & Manik, 2002), and gravity models which address the interaction between the elements of urban systems by using a similar formulation to the Newton's law of gravity (e.g. Tsekeris & Stathopoulos, 2006). Also, Agent-Based Models (ABM) and Cellular Automata (CA) have become popular for representing the actions, behavior and interactions of individual agents in space and time (Batty, 2009). In recent years, ABM and CA techniques have been particularly useful in modeling urban expansion (He, Okada, Zhang, Shi, & Zhang, 2006; Zhang, Zeng, Bian, & Yu, 2010).

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The advantage of hedonic house price model is that it allows exploring characteristics of urban systems. The market value of tradable goods, such as house prices, is linked to measurable attributes of the good being valued, thus providing a description of the urban system. The model can be estimated by means of econometric techniques such as ordinary least squares (OLS). In the case of housing market, price of properties are regressed on a set of socioeconomic, locational, and structural attributes which are generally measured at small areas such as districts or municipalities. The relationship between such attributes (explanatory variables) and house prices (response variable) correspond to the model's parameter estimates and can be interpreted as the consumer's willingness to pay for one additional unit of the corresponding house's attribute. Most of the explanatory variables generally employed in modeling house prices correspond to what Wegener (1994) and Liu (2008) have classified as urban subsystems, such as: housing (structural characteristic of properties), land use, employment, population density and location of workplaces among others. Accordingly, hedonic house price modeling can provide valuable information on the dynamics and complexity of urban systems by quantifying and relating a set of urban subsystems to a response variable. Numerous applications of house price models can be found in the literature. In particular, examples of how hedonic house price models can be used to analyze key variables in urban modeling are given by Nelson (1978), Bender and Hwang (1985), Ottensmann, Payton, and Man (2008).

All of the above mentioned methods have been exclusively used in urban studies to either characterize a current situation by modeling, or to predict or simulate future scenarios based on data from today's situation. This is what in the literature has been denoted as the *forward problem* (Scales & Snieder, 2000), that is, current data is used to fit a model from which predictions or simulations are derived. In a more recent study, Grêt-Regamey and Crespo (2011) propose the use of an *inverse problem* approach (Ashter, Borchers, & Thurber, 2005; Scales & Snieder, 2000; Tarantola, 2005) for planning sustainable urban systems. As opposed to the forward problem, in the inverse problem approach, a set of model's parameters characterizing a system are derived from a *given value* for the model's response. In urban planning, such *given value* can be defined by stakeholders as a desire future state for an urban system. Thus, inverse problem approach is intended to shift the focus in urban planning from mostly forecasting future states to planning from a future vision.

The paper is divided into two main sections: Firstly, we provide a theoretical framework to the inverse problem approach for urban planning proposed by Grêt-Regamey and Crespo (2011). We focus on system identification's tools to formulate and solve inverse models for urban systems. Secondly, we illustrate the systematic approach in a metropolitan area in Switzerland using a hedonic house price model showing how to deal with increasing population density.

The inverse approach

In inverse modeling, one deals with concepts and definitions frequently used in mathematical and econometric analysis. Yet, same expressions used by the different communities often refer to different concepts. For example, for mathematicians the term *parameter* refers to any type of quantity that defines certain characteristics of a system such as variables, constants or parameter estimates. While for econometricians *parameters* refer exclusively to linear quantities relating a dependent with the independents variables. Since we perform an econometric analysis in this study, and in order to avoid any confusion throughout the text, we will use the notation β to refer to the classical model parameters defined by econometricians, while θ will be used to denote the rest of parameters defining the model whether they are linear or not.

System identification

System identification is concerned with the formulation and estimation of mathematical models from observed input and output data. In the context of the inverse problem, we present a system identification procedure based on Pajaonk's (2009) contribution (Fig. 1). The system is defined as follows:

where the *input* (X) corresponds to quantities that influence other entities in the system through their relations to them and by this influence the system as a whole. These types of quantities are regarded as *independent variables*. The *output* (d) corresponds to measurable variables that are determined by both the *input* and the system itself. These *output* variables are also denoted as *dependent variables*. Similarly, *disturbances* (ε) are a type of input variable whose values cannot be chosen freely and follow a random probabilistic distribution. Finally, the *process* (**G**) corresponds to the transformation of input quantities to output variables. In econometrics, the process is given by the functional form relating the observed value of the dependent variable to the observed values of the independent variables through unknown parameters which can be estimated by statistical methods.

Pajonk (2009) and Tarantola (2005) argue that the scientific procedure for the identification of a system can be divided into the following three steps:

i) *Parameterization of the system*: Find a minimal set of model parameters and variables whose values completely characterize the system.
ii) *Forward modeling*: Use of a mathematical formulation to simulate the system output given values for the model parameters and the input.
iii) *Inverse modeling*: Obtain actual values of the model parameters given some values for the output of the *real* system.

If the system is well-quantified and system parameters are given based on either prior knowledge or statistical estimation, the forward model can be used for a physical system to simulate a new system output for a new set of system inputs. In contrast, the inverse model requires more sophisticated mathematical techniques to be solved as most of the inverse problems are typically ill-posed, that is, the solution of the problem is unstable or not unique.

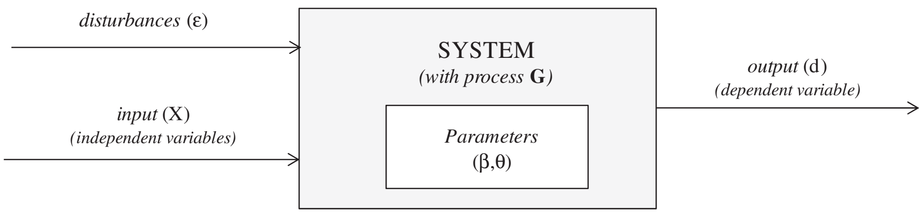

Fig. 1. Framework for describing a system defined as processes with input, disturbances, and output.

Formulation of the inverse problem

Based on Ashter et al. (2005), Tarantola (2005) and Pajonk (2009), we formulate the inverse model as follows:

$$d = \mathbf{G}(X, \beta, \theta, \varepsilon) \tag{1}$$

where d is a vector of outputs of the system, X is a matrix of input variables, β and θ are unknown parameters to be estimated, ε is a vector of unknown disturbances to be estimated. In turn **G** is a mathematical operator relating the outputs and the inputs of a system through the model's parameters. As such, **G** can take many forms, as ordinary differential equations (ODE) or partial differential equations (PDE) (Ashter et al., 2005). **G** can also be understood as a matrix representing the *process* of the system as described in Fig. 1. The set (X,β,θ,ε) can be solved for a given value of the vector of outputs *d* by inverting the model in (1) as:

$$(X, \beta, \theta, \varepsilon) = \mathbf{G}^{-1}(d) \tag{2}$$

Two points are worth noting from Equation (2). Firstly, though the value of the input variables (X) may be known a priori in many cases, new values for X can be derived from the inverse approach for a desired value of the output vector d. Secondly, Equation (2) only makes sense if the number of input variables and the number of unknown parameters are identical, i.e., **G** is a square matrix (Tarantola, 2005).

The interpretation of the solution set (X,β,θ,ε) depends not only on the phenomenon being modeled but also on the motivation of the modeling itself. Engl, Hanke, and Neubauer (2000) argue that there are two different motivations for studying inverse modeling. First, one wants to know the past states or parameters of a physical system (this is the case when parameters estimates represent the initial conditions of the system under study). Second, one wants to find out how to influence a system via its present state or via parameters in order to *steer* it to a different state in the future. Accordingly, inverse modeling has been of great contribution to applied sciences in fields such as medical imaging (Arridge, 1999; Courdurier, Noo, Defrise, & Kudo, 2008; Louis, 1997) engineering applications (Martinez-Luaces, 2009; Schneider, Mossi, Franca, de Sousa, & da Silva Neto, 2009; Soemarwoto, Labrujere, Laban, & Yanshah, 2009), geophysical applications (Menke, 1989; Parker, 1994; Scales & Tenorio, 2001; Tarduno, Bunge, Sleep, & Hansen, 2009).

Ill-posed problem

As stated in point 2.1, most inverse problems are *ill-posed*. This is a mathematical concept referring to problems in which the solution may not be unique or it is not stable under perturbations of the data. In the latter case, it is said that the solution does not depend continuously on the data, thus small changes in the observed data vector (d) may result in very different values for solution set (X,β,θ,ε). These two undesirable properties make inverse model difficult to solve. Methods for dealing with unstable solutions in ill-posed problems are called *regularization* methods. They basically consist of finding an approximation to the exact solution by using a family of neighboring well-posed problems. A comprehensive study of different regularization methods for solving linear and non-linear ill-posed inverse problems is given by Engl et al. (2000).

Another alternative to cope with ill-posed inverse problems is to constrain the space of solutions by using prior information on the phenomenon being modeled. In this respect, Engl et al. (2000) point out that the study of inverse problems involves the question of how to enforce uniqueness by additional information or assumption on the data. Similarly, Rabino and Lagui (2002) stress the importance of possessing prior information about the system in question derived from own experiences. Prior information can be employed to delimit the space of possible solutions of the problem as well as to contribute to reduce the instability of solutions. In a same way, Scales and Snieder (2000) argue that for practical inverse problems one is interested in patterns that can be used in a meaningful way for making decisions. In practice, these decisions are usually not based exclusively on the estimated model, but involve the integration of other data as well as human expertise.

In this study, we follow the above mentioned guidelines to cope with ill-posed problems by which the space of possible solutions is constrained. To this end, we use a *gray box* modeling approach proposed by Pajonk (2009) and Jones, Watton, and Brown (2007). A *gray box* approach is a type of model identification in which prior information on parameter values is employed to tackle the problem of non-unique or/and unstable solutions. Such prior information may be provided as a Bayesian prior probability distribution of parameters or as a more bounded solution interval for all terms from the set (X,β,θ,ε). In addition, some model parameters can be independently estimated by an appropriate mathematical method and then used as such in the inverse modeling. By doing this, the space of solution is significantly reduced as some parameters of the system remain fixed when solving the inverse problem. If we intend to solve the inverse problem for the input set of variables X, we can first estimate the parameters and the disturbance term of the system by an appropriate statistical method and subsequently solve the inverse problem for the input variables. Thus, Equation (2) becomes:

$$(X) = \mathbf{G}^{-1}(d, \hat{\beta}, \hat{\theta}, \hat{\varepsilon}) \tag{3}$$

where $(d, \hat{\beta}, \hat{\theta}, \hat{\varepsilon})$ correspond to the parameters and the disturbance of the model estimated by appropriated statistical methods. Therefore, we focus our motivation on solving the inverse model based on the second statement given by Engl et al. (2000), namely, to find out how to influence the output of a system via the input variables in order to *steer* it to a different state in the future.

The case study – planning for increasing population density in a metropolitan area

The Canton of Zurich (Switzerland) enjoys a prosperous and dynamic economic activity along with one of the highest quality of living in the world. The Canton comprises 171 municipalities with an approximate population of 1.3 million inhabitants. The most populated municipality is Zurich City, which comprises about 30% of the whole Canton's population. Important economic activities are the banking sector, insurance companies and manufacturing enterprises. Over the last decade, a strong continuous increase in demand for new residential buildings has been observed. In fact, the Canton of Zurich is nowadays characterized by its high population density where empty dwellings are rare and the pressure to build new dwellings is high. One of the raising concerns is how can the negative effects associated with urban densification be mitigated in order to preserve the high quality of life of such an important urban area. In the next section, we show how inverse modeling can be used to develop solutions to house price devaluations in an urban densification process.

Description and identification of the system: parameterization of the system, forward and inverse modeling

As the *process* of the underlying urban system, we use a hedonic house price model applied in Grêt-Regamey and Crespo (2011) which is based on the model employed by Loech and Axhausen

(2010), and by Loech (2010). The econometric functional form of this process is shown in Equation (4) in which the net asking rent of a property P_i (*output*), measured in Swiss francs at a location i, is regressed on a set on explanatory variables (*input*).

$$P_i = \beta_0 + \beta_1 SQM + \beta_2 ISHOUSE + \beta_3 BUILUNTI20 + \beta_4 BUIL21TO30 + \beta_5 BUIL81TO90 + \beta_6 BUIL91TO06 + \beta_7 \log(CARTT\_CBD) + \beta_8 PTACC + \beta_9 \log(RAILSTATION) + \beta_{10} HIGHWAY + \beta_{11} AIRNOISE + \beta_{12} HOTREST\_JOBS + \beta_{13} POP\_DENS + \beta_{14} FOREIGNERS + \beta_{15} TAXLEVEL + \beta_{16} SLOPE + \beta_{17} VIEW\_LAKE + \beta_{18} SOLAR\_EVE + \varepsilon_i \quad (4)$$

The independent variables are categorized as (*i*) *structural variables*: floor area (SQM), type of property (ISHOUSE), and date of construction (BUILTUNTI20 and BUIL..TO..), (*ii*) *socioeconomic variables*: number of jobs in hotels and restaurants in the proximity (HOTREST_JOBS), population density (POP_DENS), percentage of foreigners (FOREIGNERS), and tax level of the zone (TAXLEVEL), and (*iii*) *locational variables*: Average travel time to the Zurich Central Business District by car in minutes (CARTT_CBD), regional public transport accessibility to employment (PTACC), distance to the next rail station (RAILSTATION), presence of a highway in the proximity (HIGHWAY), presence of noise level above 52 dB (AIRNOISE), terrain slope (SLOPE), visibility of the lake (VIEW_LAKE), and evening solar exposure (SOLAR_EVE). Finally, β denotes a *parameter* to be estimated, ε is a random error (*disturbance*) term to be estimated, which is assumed to be independently and identically distributed. A more detailed description of all variables is given in Appendix A.

To estimate the set of β parameters, the hedonic model is calibrated using data obtained for 8541 geocoded properties through various Swiss real estate online platforms between December 2004 and October 2005. The addresses for all dwelling units in the dataset were geocoded at building level and matched with a wide set of spatial variables. The model was first calibrated using ordinary least squares regression to produce the beta parameter estimates ($\hat{\beta}$) reported in Table 1. The model fitting showed a high goodness-of-fit with and R^2 of 0.77. A Moran's test was performed to check the spatial autocorrelation of residuals. The test showed a statistically significant but rather low spatial autocorrelation among residuals, with a Moran's *I* index (based on the inverse of the square distance between the locations) equals 0.16. Also, the negative sign of the POP_DENS parameter estimate indicates that house prices are negatively related to population density. In the same way, the negative sign of the FOREIGNERS parameter estimate shows that house prices drop as the proportion of foreigners becomes larger in the area.

Table 1
OLS parameter estimates.

Variable	Parameter estimates	Standarized parameter estimates	*t*-value
SQM	18.9	0.77	128.4*
ISHOUSE	290	0.05	8.5*
BUILBEF20	106.6	0.04	6.1*
BUIL21TO30	166.7	0.03	5.6*
BUIL81TO90	−36.4	−0.01	−2.7**
BUIL91TO06	50.8	0.02	3.8*
Log(CARTT_CBD)	−697.4	−0.27	−36.3*
PTACC	20.5	0.04	5.8*
Log(RAILSTATION)	−48.5	−0.04	−7.4*
AUTOBAHN	−128.7	−0.02	−3.7*
AIRNOISE	−85.2	−0.03	−5.9*
HOTREST_JOBS	0.059	0.05	7.5*
POP_DENS	−0.708	−0.06	−9.9*
FOREIGNERS	−628.9	−0.03	−4.6*
TAXLEVEL	−2.37	−0.04	−6.3*
SLOPE	1052	0.04	6.32*
VIEW_LAKE	0.077	0.09	15.3*
SOLAR_EVE	1073	0.10	17.7*

R-square = 0.77.
Moran's *I* = 0.16*.
*: significant at 0.00% level.
**: significant at 0.01% level.

In order to investigate how different mitigation schemes for the densification problem vary over space, we calibrated Equation (4) using a mixed geographical weighed regression (GWR) (Fotheringham, Brunsdon, & Charlton, 2002). The functional form of the mixed-GWR used in this study is represented as:

$$P = X_a\beta_a + X_b\beta_b + \varepsilon \quad (5)$$

where P is a vector of the monthly rents of the property, X_a and X_b is the matrix of explanatory variables associated with global and local coefficients respectively, β_a are the global coefficients (without including the intercept term), β_b is a matrix of location-specific coefficients (including the intercept term), and ε is a vector of residuals assumed to be random and spatially uncorrelated. In this study, we follow the mixed-GWR model used by Grêt-Regamey and Crespo (2011) in which global and local variables are classified based on an exploratory data analysis of the independent variables. Thus, variables exhibiting low spatial variability are likely to produce spurious results in local models so that they are classified as global. Further methods for testing the spatial non-stationarity of parameters can be found in Fotheringham et al. (2002) and Leung, Mei, and Zhang (2000). In our model, the global variables include PTACC, RAILSTATION, HIGWHAY, AIRNOISE, SLOPE, TAXLEVEL, VIEW_LAKE, and SOLAR_EVE. The local variables include SQM, ISHOUSE, BUILTUNTI20, BUIL..TO.., CARTT_CBD, HOTREST_JOBS, POP_DENS, and FOREIGNERS.

Compensation scheme by inverse modeling

In order to explore the effects of increasing urban population density on house prices in a statistically meaningful way, we select seven clusters from the sampled locations to perform the densification analysis. As our intention is to produce spatially explicit results, points of each cluster are selected and grouped together in a way that the distance to points of the nearest cluster ranges between 10 and 15 km. In this manner, each cluster is more likely to represent a different submarket in the Zurich housing market. As can be seen in Fig. 2, cluster 5 is located in Zurich City which is the most populated city in the Canton and probably the most attractive city for future inhabitants. Cluster 3 corresponds to Winterthur, the second most populated city in the Canton, and also a city that has been enjoying and increasing economic activity over the last decades. Cluster 6 is located in one of the best-off residential areas in the Canton of Zurich. In fact, this cluster can be used in the analysis as a representative example of most properties located alongside the lake of Zurich (the biggest lake in Fig. 2). In turn, clusters 1, 2, 4, and 7 represent various popular areas for residents who commute to work in Zurich City. Note that the number, spatial arrangement, and size of clusters are additionally subject to the number and spatial distribution of the sampled data. A summary of the averaged explanatory variables for each cluster is presented in Appendix B.

At a cluster level, the average of house rent estimates ($\bar{\hat{P}}^k$) is obtained as:

$$\bar{\hat{P}}^k = \bar{X}_a^k\hat{\beta}_a + \bar{X}_b^k\bar{\hat{\beta}}_b^k \quad (6)$$

where k:1,...7 denotes the index of the cluster, $\bar{X}_a^k$ and $\bar{X}_b^k$ are matrices of averaged explanatory variables associated with global

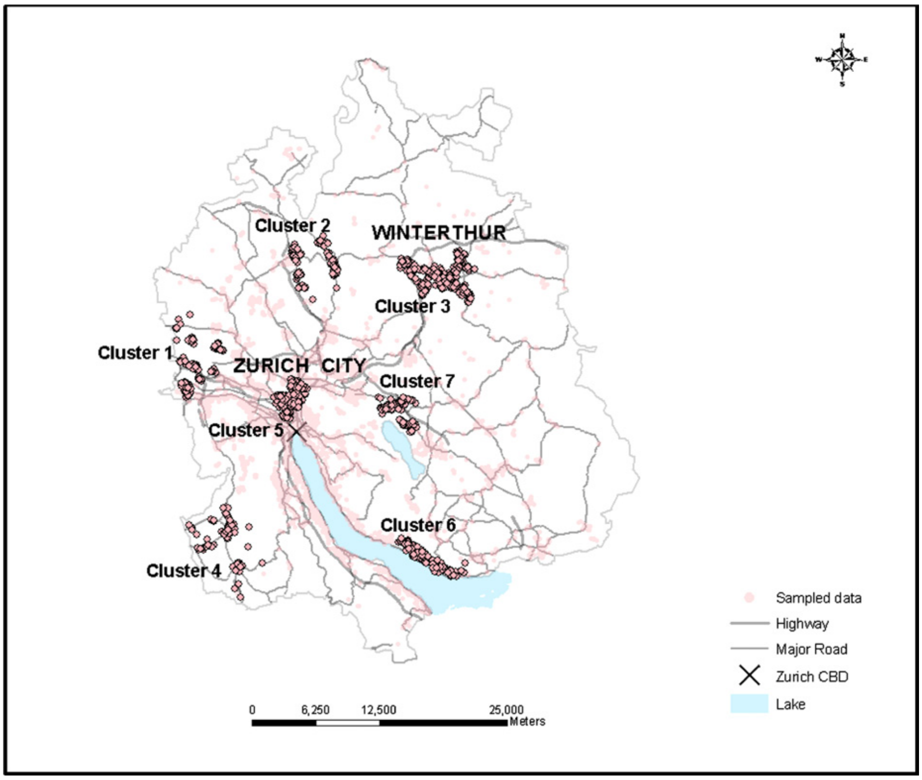

Fig. 2. Spatial distribution of sampled locations and selected clusters for the inverse analysis.

and local variables respectively for cluster k, $\hat{\beta}_a$ is the vector of global parameter estimates derived from the OLS regression (see Table 1), and $\bar{\hat{\beta}}_b^k$ is the vector of averaged mixed-GWR local estimates for cluster k (A summary of the averaged local estimates for each cluster is presented in Appendix C). We omit the estimated error term $\hat{\varepsilon}$ from the analysis as our motivation is to model the expected value (the trend) of house rents for a given set of explanatory variables.

The next step is to solve the *inverse problem* so as to find the compensation schemes for the devaluation of house rents caused by a densification. Since the Canton of Zurich is nowadays enjoying a prosperous and dynamic activity we select a select a densification scheme in which the population density and the proportion of foreigners increase by 25% and 20% respectively with respect to the current cluster levels. This level of densification may occur gradually in a lapse of 5, 10 or 20 years. For illustration purposes, we select two explanatory variables to investigate the compensation scheme associated with the selected level of densification: the average travel time to the Zurich CBD by car in minutes (CARTT_CBD) and the income tax level (TAXLEVEL). While the CARTT_CBD variable is associated with the development of transport infrastructures of cities, the TAXLEVEL variable is closely related to public policy affairs. Next, we follow the inverse approach shown in Equation (3) in which $\hat{\theta}$ becomes zero according to the formulation of the hedonic modeling shown in Equation (4). Thus, Equation (3) can be re-written at a cluster level for the hedonic house price model as:

$$(\overline{X}^k) = \mathbf{G}^{-1}\left(P_d^k, \hat{\beta}_a, \bar{\hat{\beta}}_b^k\right) \quad (7)$$

where $\overline{X}^k$ is a vector representing the average of the explanatory variables for cluster k, $\hat{\beta}_a$ and $\bar{\hat{\beta}}_b^k$ are as defined in Equation (6), and P_d^k is the desired house rent level to be achieved for cluster k. In other words, P_d^k corresponds to the averaged house rent plus the drop in the house rents caused by densification for cluster k. Fig. 3 shows the percentage by which the averaged house rent drop as a result of the chosen densification scheme.

Subsequently, we solve Equation (7) for the two above mentioned explanatory variables in order to compensate the drop in the house rent depicted in Fig. 4. Results for the compensation through the average travel time to the Zurich CBD in minutes by car are presented in Fig. 4 which depicts by how much such variable has to be reduced to compensate the devaluation of house rents according to the selected densification scheme. Results show significant spatial variability across the seven clusters. For example, residents from cluster 4 and 5 respond much more slowly to variation in the accessibility to the Zurich CBD by car than residents from the other clusters. This may be explained by the fact that the workplace of most residents from both clusters is not located in the Zurich CBD or/and because such residents prefer to use public transportation to get to the Zurich CBD instead of the car. Conversely, residents from clusters 2 and 7 respond fairly rapidly to variation in the accessibility to the Zurich CBD which suggests that the workplace of most of such residents is located in the Zurich CBD.

Similarly, Fig. 5 depicts by how much the current tax level has to be reduced to compensate the devaluation of house rents according to same densification scheme. It has to be noted here, that the parameter estimate associated with the TAXLEVEL variable corresponds to a global parameter in the model, thus the spatial variability observed in Fig. 5 is explained by the extent to which house rents are devaluated as a result of densification through the local variables population density and percentage of foreigners alone. Residents from clusters 4, 5, 6 and 7 respond more slowly to changes in the tax level than residents from the

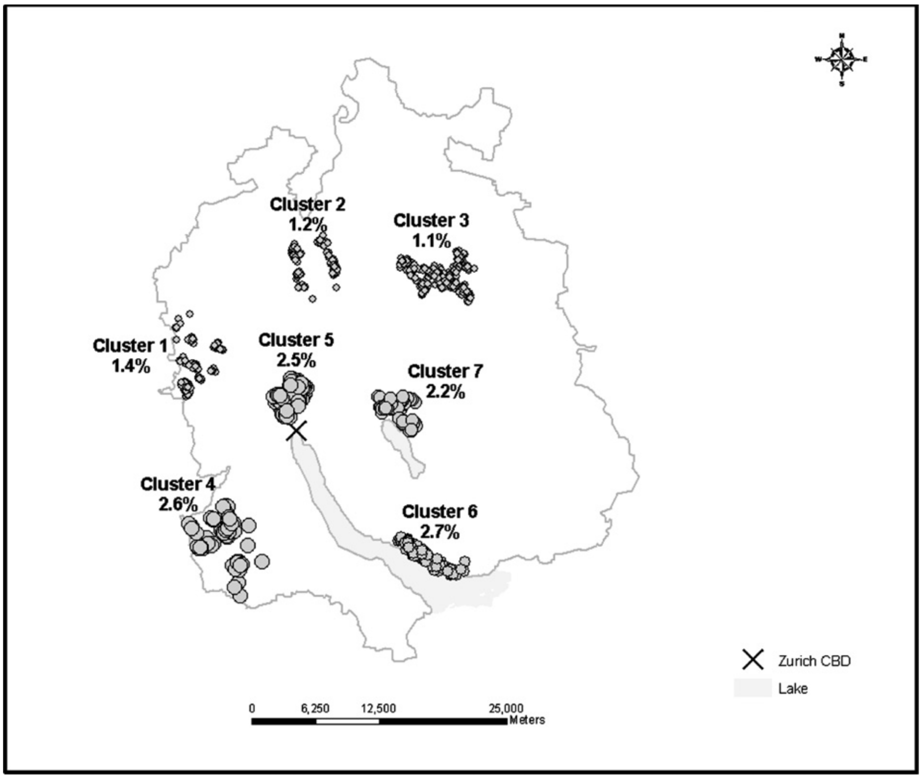

Fig. 3. Percentage decrease in house rents caused by an increase of 25% in population density and 20% in proportion of foreigners. The size of the symbols is proportional to the reduction.

other clusters, i.e., the percentage by which the income tax level must be reduced to compensate devaluation of house rents is higher for these clusters than for the rest. This can be explained because the "cost" of densification tends to be higher for areas densely populated, such as cluster 5 and 7, as the demand for housing is likely to exceed the supply. In a similarly way, densification processes are frequently difficult in areas sparsely populated, such as clusters 4 and 6, as existing residents often become reluctant to possible negative effects of densification on the their quality of living. The only exception is cluster 3 for which a relative low reduction in tax level is needed to compensate the cost of densification despite its relatively high population density indicating that cluster 3 might be a potential area for future densification.

Trade-off between two compensation variables

Alternative to the compensation scheme used in Section 4, Equation (7) can be solved simultaneously for the accessibility to the Zurich CBD by car and the income tax level variables. In this way, a trade-off between both variables can be used to compensate the devaluation of house rents. The trade-off for each cluster is illustrated in Fig. 6. As with the previous example, Fig. 6 is based on the densification scheme in which the population density increases by 25% and the proportion of foreigners by 20% with respect to the current cluster levels. For example, in the case of cluster 4, a reduction of 10 min in the average time to the Zurich CBD by car is needed to compensate the cost of densification. However, if simultaneously the tax level is reduced by about 12.5%, only a 3 min reduction in the accessibility to the Zurich CBD by car is needed to compensate the densification cost. From an urban planning perspective, a reduction of 10 min in the accessibility to a particular place by car becomes technically unfeasible unless enormous changes in the transport infrastructure are done. Similarly, with a reduction of about 2 min in the average travel time from cluster 6 to the Zurich CBD by car in case of cluster 6, the reduction in the tax level needed to compensate the cost of densification becomes about 11% instead of the original 23.8%. Thus, the trade-off analysis allows us to replace some unfeasible compensation schemes by a combination of more achievable compensation scheme. Finally, it is also worth noting that Fig. 6 is also a representation of the ill-posed problem in inverse modeling, as each line corresponds to a set of possible combinations of the two selected explanatory variables leading to the same model result. Arguably, the ill-posed problem is a condition for the trade-off analysis.

Final remarks

Inverse modeling has long been used in various science fields including medicine, physics and engineering, however, not without some difficulties caused mostly by the ill-posed problem. Strategies to cope with the ill-posed problem are dependent on the system process and the system parameterization. Therefore, appropriate system identification becomes an essential step prior attempting to find a solution. In this study, we have provided a theoretical framework to support planners in using the inverse approach for urban studies. We suggest the use of *regularization* methods as a way to constrain the space of possible solutions, reducing in this way the instability of the inverse problem's solutions. Particularly, we present the *gray box* approach by which some parameters of the system are estimated using econometric techniques prior finding the solutions of the inverse problem.

In the case study, we show how the proposed method operates and how it can be used to support urban decision-making processes. In particular, the use of a hedonic house price model

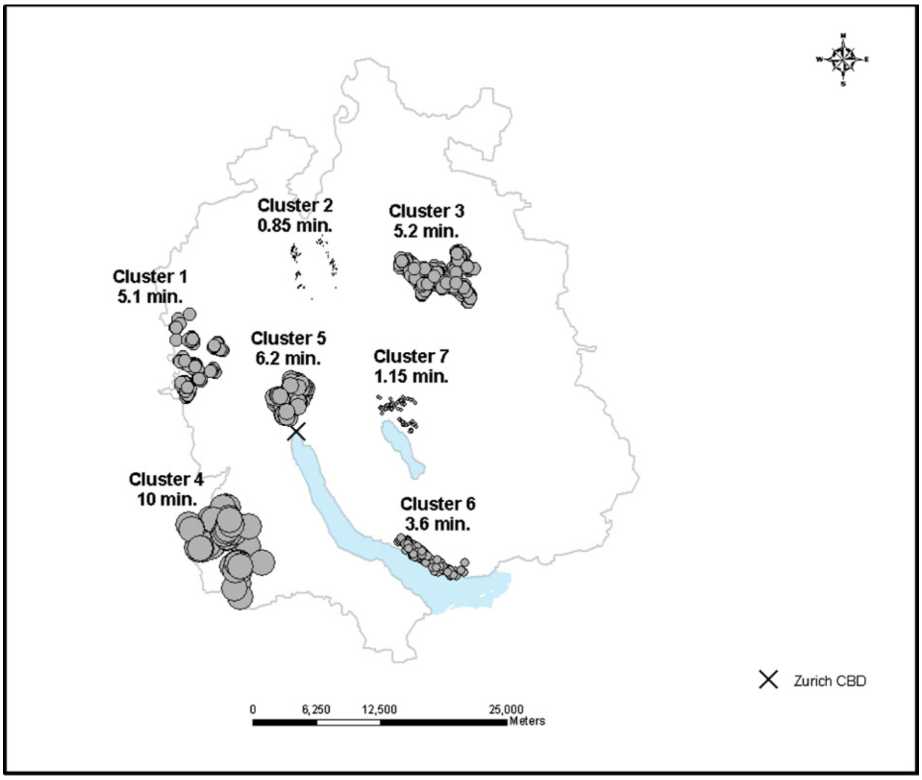

Fig. 4. Reduction in the average time to Zurich CBD by car in minutes to compensate an increase of 25% in population density and 20% in percentage of foreigners. The size of the symbols is proportional to the reduction.

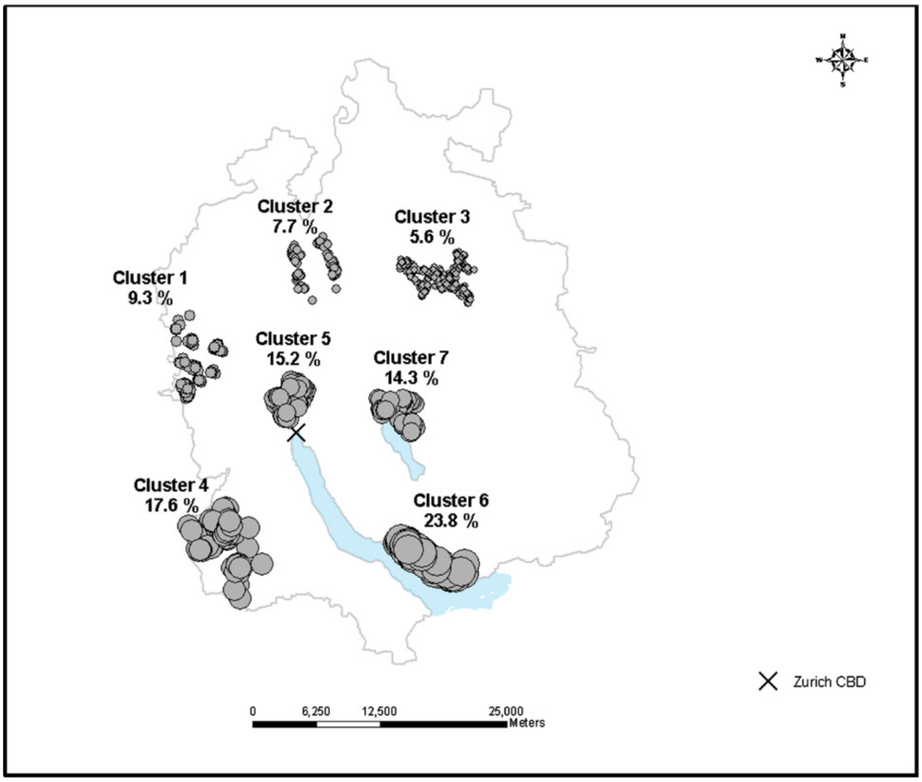

Fig. 5. Percentage reduction in the income tax level to compensate an increase of 25% in population density and 20% in foreigners. The size of the symbols is proportional to the reduction.

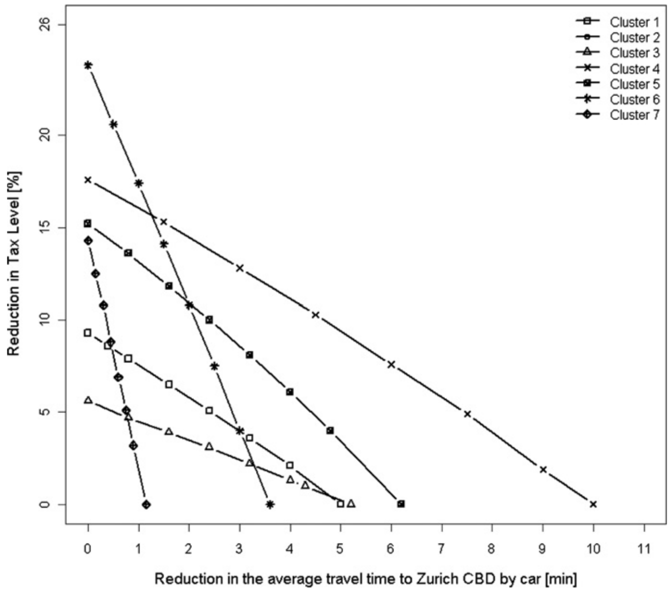

Fig. 6. Trade-off between tax level and accessibility to the Zurich CBD by car.

allowed us to show how intrinsic problems related to urbanization processes, such as densification, may negatively affect the economic value of properties in a metropolitan area. In this case, the desired outcome for the urban system under study must be such that the economic loss caused by densification is compensated by new levels of key variables chosen from a set of house determinants. In this study, we select two variables to illustrate the power of the inverse approach with the densification problem: the income tax level and the accessibility to the Zurich Central Business District (CBD) by car. Of particular interest is the trade-off analysis of possible solutions shown in Fig. 6, because this is a representation of how the ill-posed problem can also be used in favor of urban planning.

As a way to minimize the effects of the statistical uncertainty associated with the model estimates on the inverse model analysis, we perform the compensation analysis at cluster level and not a single building level. Averaged values of estimates reduced the inherent parameters uncertainty as extreme values are down-weighted so that the bias of predicted values is reduced. However, it is important to note that the accuracy of the results is also dependent on the size of clusters. In general terms, it is argued that the bigger the size of the cluster, the higher the biased of the averaged local estimates employed in the analysis. The statistical accuracy of the results is also subject to the spatial location of the selected clusters as the goodness-of-fit of the econometric technique employed in the analysis (mixed-GWR) is most likely to vary over space. The mixed-GWR estimates tend to be more accurate in areas where the sample data is more densely distributed than in areas where data is scarcer.

In conclusion, despite these statistical limitations, this study has shown that inverse modeling is a powerful tool for developing spatial plans utilizing future visions, especially if multiple variables are employed to carry out trade-off analyses In fact, the more variables are employed in the trade-off analysis, the broader the space of possible solutions of the model becomes, facilitating in this way the selection of alternative and feasible solutions for planners. In addition, it is also important to note that inverse approach in urban modeling is not necessarily subject to a specific planning problem or mathematical model and so other statistical techniques than hedonic modeling can be used in future research. For example, inverse modeling can be used as an alternative or complementary approach to traditional logistic land use change models (Bakker et al., 2005; Verburg, van Eck, de Nijs, Dijst, & Schot, 2004;) as well as to traditional spatial econometric models employed in regional converge studies as documented in Rey and Montouri (1999).

Further research may also address the inverse modeling approach using spatio-temporal data. In such case, results of the inverse model might account for the initial conditions of the urban system evolving over time. Finally, it is worth pointing out that inverse model approach can also be used in conjunction with backcasting (Dreborg, 1996; Holmberg, 1998). For example, while backcasting can be employed to identify processes and important steps to reach desired future scenarios in sustainable urban planning (Grêt-Regamey & Brunner, 2011), inverse modeling can be used to quantify the feasibility of various developing alternatives in order to reach such scenarios.

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Appendix A. Descriptive statistics of the model's variables

Variable	Type[a]	Description	Min	Max	Mean
Response variable					
RENT	C	Monthly net asking rent in CHF	476	15,000	1841
Explanatory variables					
1.- Structural variables					
SQM	C	Floor area in square meter	20	400	91.3
ISHOUSE	D	Is 1 if the property is a single family house, 0 otherwise.	0	1	
BUILBEF20	D	Is 1 if the property was built prior 1920, 0 otherwise.	0	1	
BUIL21TO30	D	Is 1 if the property was built between 1921 and 1930.	0	1	
BUIL31TO80[b]	D	Is 1 if the property was built between 1931 and 1980.	0	1	
BUIL81TO90	D	Is 1 if the property was built between 1981 and 1990.	0	1	
BUIL91TO06	D	Is 1 if the property was built between 1991 and 2006.	0	1	
2.- Locational variables					
CARTT_CBD	C	Average travel time to the Zurich Central Business District by car in minutes.	8	58.4	29.9
PTACC	C	Regional public transport accessibility to employment.	−19.4	12.4	10.7
RAILSTATION	C	Euclidean distance to next rail station in km.	0.013	5.732	0.911
HIGHWAY	D	Is 1 if highway located within 100 m, 0 otherwise.	0	1	
AIRNOISE	D	Is 1 is daily average air noise is above 52 dB, 0 otherwise	0	1	
SLOPE	C	Slope by 25 m raster	0.00	0.26	0.036
VIEW_LAKE	C	Visibility[c] of lake surface (>1 sqkm) in hectares	0.0	441.8	8887.8
SOLAR_EVE	C	Evening solar exposure index	0.0	607.1	238.2
3.- Socioeconomic variables					
HOTREST_JOBS	C	Number of jobs in hotel and restaurant industry within 1 km.	1.2	7083.8	314.0
POP_DENS	C	Number of inhabitants in hectare	1.0	2004.0	93.5

(*continued*)

Variable	Type[a]	Description	Min	Max	Mean
FOREIGNERS	C	Percentage of foreigners[d] in hectare	0.00	0.50	0.05
TAXLEVEL	C	Local income tax level as percentage of the basic cantonal tax	69.0	122.0	110.3

[a] C = continuous; D = binary.
[b] Reference case, therefore this variable is not included in the calibration of the hedonic house price model.
[c] Visibility of VIEW_LAKE variable is calculated based on 25 m DEM, not considering objects such as buildings or trees.
[d] Foreigners are defined as inhabitants with nationalities outside of North-Western Europe, North America and Australia.

Appendix B. Summary of the main exploratory variables for each cluster

Cluster 1	
Number of properties selected	211
Average house price (monthly net rent in CHF)	1619
Average floor area (SQM)	89.8
Number of flats (ISHOUSE = 0)	211
Number of properties built prior 1920 (BUILBEFO20)	12
Number of properties built between 1921 and 1930 (BUIL21TO30)	2
Number of properties built between 1931 and 1980 (BUIL31TO80)	108
Number of properties built between 1981 and 1990 (BUIL81TO90)	42
Number of properties built between 1991 and 2006 (BUIL91TO06)	47
Average travel time to the Zurich CBD by car in minutes (CARTT_CBD)	33.4
Regional public transport accessibility to employment (PTACC)	10.8
Visibility of lake surface (VIEW_LAKE)	0
Average tax level (TAXLEVEL)	106.8
Average number of jobs in hotel and restaurant industry within 1 km (HOTREST_JOBS)	91
Average population density in hectare (POP_DENS)	93.5
Average proportion of foreigners in hectare (FOREIGNERS)	0.051
Cluster 2	
Number of properties selected	233
Average house price (monthly net rent in CHF)	1718
Average floor area (SQM)	98.5
Number of flats (ISHOUSE = 0)	233
Number of properties built prior 1920 (BUILUNTI20)	12
Number of properties built between 1921 and 1930 (BUIL21TO30)	0
Number of properties built between 1931 and 1980 (BUIL31TO80)	88
Number of properties built between 1981 and 1990 (BUIL81TO90)	63
Number of properties built between 1991 and 2006 (BUIL91TO06)	70
Average travel time to the Zurich CBD by car in minutes (CARTT_CBD)	36.3
Regional public transport accessibility to employment (PTACC)	9.8
Visibility of lake surface (VIEW_LAKE)	0
Average tax level (TAXLEVEL)	108.7
Average number of jobs in hotel and restaurant industry within 1 km (HOTREST_JOBS)	117.4
Average population density in hectare (POP_DENS)	83.9
Average proportion of foreigners in hectare (FOREIGNERS)	0.042
Cluster 3	
Number of properties selected	421
Average house price (monthly net rent in CHF)	1473
Average floor area (SQM)	83.6
Number of flats (ISHOUSE = 0)	421
Number of properties built prior 1920 (BUILUNTI20)	40
Number of properties built between 1921 and 1930 (BUIL21TO30)	12
Number of properties built between 1931 and 1980 (BUIL31TO80)	246
Number of properties built between 1981 and 1990 (BUIL81TO90)	54
Number of properties built between 1991 and 2006 (BUIL91TO06)	69
Average travel time to the Zurich CBD by car in minutes (CARTT_CBD)	43.1
Regional public transport accessibility to employment (PTACC)	10.7
Visibility of lake surface (VIEW_LAKE)	0
Average tax level (TAXLEVEL)	122
Average number of jobs in hotel and restaurant industry within 1 km (HOTREST_JOBS)	242.8
Average population density (POP_DENS)	108.9
Average proportion of foreigners (FOREIGNERS)	0.05

(*continued*)

Cluster 4	
Number of properties selected	180
Average house price (monthly net rent in CHF)	1877
Average floor area (SQM)	106.7
Number of flats (ISHOUSE = 0)	180
Number of properties built prior 1920 (BUILUNTI20)	19
Number of properties built between 1921 and 1930 (BUIL21TO30)	0
Number of properties built between 1931 and 1980 (BUIL31TO80)	40
Number of properties built between 1981 and 1990 (BUIL81TO90)	53
Number of properties built between 1991 and 2006 (BUIL91TO06)	68
Average travel time to the Zurich CBD by car in minutes (CARTT_CBD)	39.0
Regional public transport accessibility to employment (PTACC)	10.4
Visibility of lake surface (VIEW_LAKE)	211.2
Average tax level (TAXLEVEL)	188.3
Average number of jobs in hotel and restaurant industry within 1 km (HOTREST_JOBS)	51.0
Average population density (POP_DENS)	61.5
Average proportion of foreigners (FOREIGNERS)	0.04
Cluster 5	
Number of properties selected	302
Average house price (monthly net rent in CHF)	1788
Average floor area (SQM)	76.2
Number of flats (ISHOUSE = 0)	302
Number of properties built prior 1920 (BUILUNTI20)	82
Number of properties built between 1921 and 1930 (BUIL21TO30)	32
Number of properties built between 1931 and 1980 (BUIL31TO80)	118
Number of properties built between 1981 and 1990 (BUIL81TO90)	54
Number of properties built between 1991 and 2006 (BUIL91TO06)	16
Average travel time to the Zurich CBD by car in minutes (CARTT_CBD)	18.4
Regional public transport accessibility to employment (PTACC)	12.0
Visibility of lake surface (VIEW_LAKE)	458.1
Average tax level (TAXLEVEL)	122
Average number of jobs in hotel and restaurant industry within 1 km (HOTREST_JOBS)	1305.8
Average population density (POP_DENS)	141.3
Average proportion of foreigners (FOREIGNERS)	0.07
Cluster 6	
Number of properties selected	195
Average house price (monthly net rent in CHF)	2024
Average floor area (SQM)	98.2
Number of flats (ISHOUSE = 0)	195
Number of properties built prior 1920 (BUILUNTI20)	28
Number of properties built between 1921 and 1930 (BUIL21TO30)	7
Number of properties built between 1931 and 1980 (BUIL31TO80)	103
Number of properties built between 1981 and 1990 (BUIL81TO90)	12
Number of properties built between 1991 and 2006 (BUIL91TO06)	45
Average travel time to the Zurich CBD by car in minutes (CARTT_CBD)	38.1
Regional public transport accessibility to employment (PTACC)	10.2
Visibility of lake surface (VIEW_LAKE)	2184.2
Average tax level (TAXLEVEL)	96.1
Average number of jobs in hotel and restaurant industry within 1 km (HOTREST_JOBS)	67.1
Average population density (POP_DENS)	56.2
Average proportion of foreigners (FOREIGNERS)	0.04
Cluster 7	
Number of properties selected	259
Average house price (monthly net rent in CHF)	1635
Average floor area (SQM)	92.5
Number of flats (ISHOUSE = 0)	259
Number of properties built prior 1920 (BUILUNTI20)	4
Number of properties built between 1921 and 1930 (BUIL21TO30)	0
Number of properties built between 1931 and 1980 (BUIL31TO80)	132
Number of properties built between 1981 and 1990 (BUIL81TO90)	52
Number of properties built between 1991 and 2006 (BUIL91TO06)	71
Average travel time to the Zurich CBD by car in minutes (CARTT_CBD)	31.3
Regional public transport accessibility to employment (PTACC)	10.8
Visibility of lake surface (VIEW_LAKE)	221.7
Average tax level (TAXLEVEL)	101.8
Average number of jobs in hotel and restaurant industry within 1 km (HOTREST_JOBS)	90.6
Average population density (POP_DENS)	105.2
Average proportion of foreigners (FOREIGNERS)	0.04

Appendix C. Averaged values local estimates for each cluster

Averaged local estimates	Cluster 1	Cluster 2	Cluster 3	Cluster 4	Cluster 5	Cluster 6	Cluster 7
SQM	13.1	13.0	13.7	13.7	20.9	17.7	12.3
ISHOUSE	561.6	−27.5	473.3	376.3	353.2	786.4	414.0
BUILUNTI20	162.2	227.1	51.2	154.6	143.6	99.1	81.2
BUIL21TO30	−56.3	664.2	115.1	263.4	10.4	128.6	−81.6
BUIL81TO90	34.7	−46.5	−13.4	97.2	−67.9	44.8	38.8
BUIL91TO06	191.8	108.6	122.6	192.2	19.7	113.5	298.0
Log(CARTT_CBD)	−142.0	−831.4	−125.6	−168.5	−107.6	−550.8	−919.3
HOTREST_JOBS	−0.29	0.26	0.25	1.05	0.075	−0.45	−0.20
POP_DENS	−0.57	−0.55	−0.14	−2.44	−0.78	−2.9	−0.84
FOREIGNERS	−1014.0	−980.8	−1229.4	−1598.7	−1239.7	−1893.0	−1664.5

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